IUT and modern number theory

Ivan Fesenko

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CFT and its generalisations

CFT = Class Field Theory, HCFT = Higher CFT, HAT = Higher Adelic Theory,

IUT is the first development which has systematic fundamental applications to Diophantine Geometry. LC (= Langlands Correspondences) have some.

LC, HAT and recently IUT have applications to Analytic Number Theory.

LC, IUT and HAT have applications to Arithmetic of Elliptic Curves.

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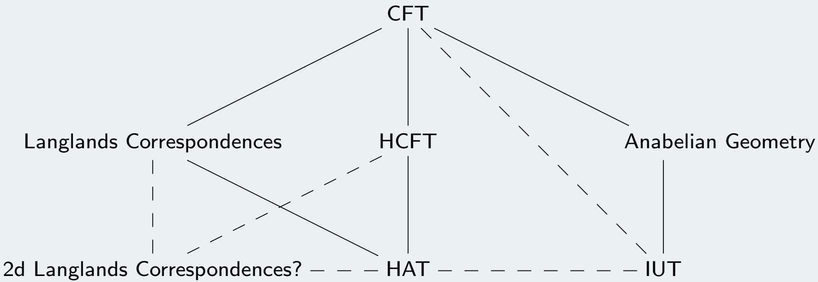

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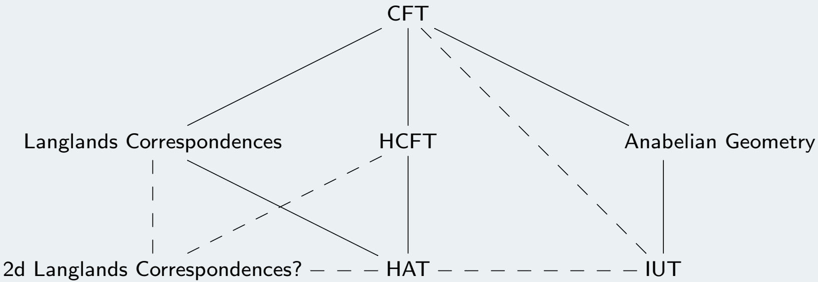

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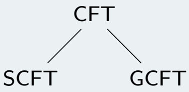

SCFT = special CFT

Using torsion/division points or values of appropriate functions at torsion points to generate finite extensions of the base fields under investigation and to describe the Galois action on them.

Cyclotomic: Kronecker, Weber, Hilbert.

Using elliptic curves with CM: Kronecker, Weber, a relevant portion of Takagi's work.

Using abelian varieties with CM: Shimura.

These theories are not extendable to arbitrary number fields. They are not functorial.

Hilbert Problem 12 was about extensions of SCFT to number fields, the best was achieved by Shimura.

Local SCFT using Lubin–Tate formal groups works over any local field with finite residue field and does not work over local fields with infinite perfect residue field.

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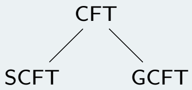

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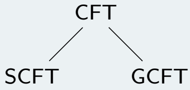

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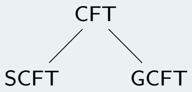

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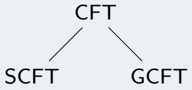

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GCFT = general CFT

These theories follow very different conceptual patterns than SCFT.

The list of GCFTs for arithmetic fields includes:

Takagi 1920, the first work in GCFT with his general existence theorem and its applications;

Artin reciprocity map, whose full construction uses Chebotarev's theorem;

Hasse, the use of the Brauer group in CFT, the first local CFT, local-to-global aspects;

Chevalley's invention of ideles, local-to-global, the global reciprocity map as the product of the local reciprocity maps, whose kernel contains the diagonal image of global elements.

Classical approaches to CFT are presented, among many sources, in Hasse's Klassenkörperbericht, and in Weil's and Lang's books.

Cohomological approaches: Artin–Tate, ...

Finding explicit formulas for the Hilbert pairing and its generalisations was one of the ways to get more explicit information about the reciprocity map.

Many of these explicit formulas involve indeterminacies.

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Post-cohomological and cohomologically-free theories: *explicit and algorithmic*,

Tate–Dwork, Hazewinkel, Neukirch, F

These theories:

◇ clarified and made explicit some of the key structures of CFT

◇ they are less dependent on torsion and they do not use the Brauer group

◇ they are explicit and algorithmic

◇ they are easy

◇ they really explain CFT.

Remark. In his explicit GCFT Neukirch was partially motivated by his work in anabelian geometry of number fields.

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CFT mechanism

CFT mechanism discovered by Neukirch.

Start with an abelian topological group A endowed with a continuous action by a profinite group G.

Think of G as the absolute Galois group G_k of a field k.

For an open subgroup G_K of G_k denote by A_K the G_K-fixed elements of A.

Denote by $N_{K/k}\colon A_K \to A_k$ the product of the action of right representatives of G_K in G_k.

Assumption 1 ($\hat{\mathbb{Z}}$ quotient of G): let there be a surjective homomorphism of profinite groups $\deg\colon G_k \to \hat{\mathbb{Z}}$.

Denote its kernel $G_{\tilde{k}}$.

Then for an open subgroup G_K of G_k we get a surjective homomorphism

$$\deg_K = |G_k : G_K G_{\tilde{k}}|^{-1} \deg_k \colon G_K \to \hat{\mathbb{Z}}.$$

Any element of G_K which is sent by $\deg_K$ to $1 \in \hat{\mathbb{Z}}$ is called a frobenius element w.r.t. $\deg_K$.

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Assumption 2 (a weak form of valuation compatible with deg): there is a homomorphism

$$v\colon A_k \to \hat{\mathbb{Z}}, \qquad v(A_k) = \mathbb{Z} \quad \text{or} \quad v(A_k) = \hat{\mathbb{Z}}$$

such that

$$v(N_{K/k}A_K) = |G_k : G_K G_{\tilde{k}}|\, v(A_k) \quad \text{for all open subgroups } G_K \text{ of } G_k.$$

Extensions of K inside $K\tilde{k}$ can be viewed as "unramified" extensions wrt $(\deg, v)$.

The pair $(\deg, v)$ defines a reciprocity map in the following way.

For a finite extension K of k and a finite Galois extension L/K and σ in its Galois group find any $\tilde{\sigma} \in G(L\tilde{k}/K)$ such that

$$\deg(\tilde{\sigma}) \in \mathbb{N}_{\geq 1} \quad \text{and } \tilde{\sigma}|_L = \sigma.$$

$\mathbb{N}_{\geq 1}$ can be viewed as a frobenius-like object inside étale-like object $\hat{\mathbb{Z}}$.

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Denote by Σ the fixed field of $\tilde{\sigma}$.

Then $L\tilde{k} = \Sigma\tilde{k}$, so the base change L/K to $L\Sigma/\Sigma$ produces an "unramified" extension, and $\tilde{\sigma}$ is a frobenius element of G_Σ.

Call $\pi_K \in A_K$ such that $|\hat{\mathbb{Z}} : \deg(G_K)|^{-1} v(N_{K/k}(\pi_K)) = 1$ a prime element of A_K.

Then π_K remains prime in all "unramified" extensions of K.

Now for the pair $(\deg, v)$ define the reciprocity map

$$\Psi_{L/K} : \sigma \mapsto N_{\Sigma/K}\pi_\Sigma \mod N_{L/K}A_L$$

where π_Σ is any prime element of A_Σ.

We have indeterminacies associated to the choice $\tilde{\sigma}$ and the choice of prime element.

If appropriate axioms for A under the action of G (axioms of CFT) are satisfied, then

◇ $\Psi_{L/K}$ is well defined, and it induces an isomorphism $G(L/K)^{\text{ab}} \to A_K/N_{L/K}A_L$,

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CFT mechanism

This CFT mechanism is purely group theoretical and does not depend on ring structures.

However, to verify the CFT axioms for local or global fields one has to use ring structures.

In local CFT of cdvf with finite residue field one takes the maximal unramified extension of $\mathbb{Q}_p$ or the maximal constant extension as $\tilde{k}/k$.

In CFT of global fields one takes the only $\hat{\mathbb{Z}}$-subextension of the maximal abelian extension of $\mathbb{Q}$ or the maximal constant extension as $\tilde{k}/k$.

Classical study of class formations aimed to derive CFT from as few axioms as possible.

The long term search for class formations can be interpreted as distinguishing purely monoid theoretical aspects of CFT (CFT mechanism) from its ring theoretical aspects (proving axioms of CFT).

CFT mechanism is similar to Kummer theory mechanism in the sense that both are purely monoid theoretical.

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Problem. Are there generalisations of the CFT mechanism which may have applications in anabelian geometry and LC?

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Some 2d objects

There are several types of data associates to an integral normal 2d scheme S flat over $\mathbb{Z}$ or $\mathbb{F}_p$ (surface):

$\diamond$ 2d global field: the function field K of S;

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$\diamond$ 2d local-global fields: the function field K_y of the completion of the local ring of a curve $y \subset S$, K_y is a cdvf with global residue field, from these objects one produces 2d adeles $\mathbf{B}$;

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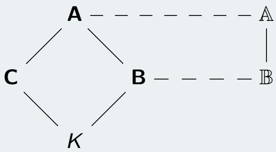
Higher adelic theory (HAT) operates with six adelic objects on surfaces:



Geometric 2d adelic structure **A** is related to rank 1 local integral structure.

Self-duality of its additive group, endowed with appropriate topology, is stronger Serre duality and it implies the Riemann–Roch theorem on surfaces.

$K_2(\mathbf{A})$ is used in HCFT.

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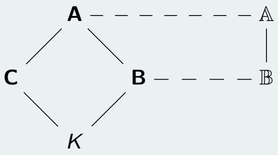

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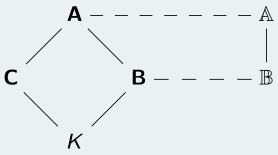

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HCFT in characteristic zero was first produced by Kato and Kato–Saito, working with higher Kummer theory for Milnor K-groups. These higher GCFT are not explicit.

A generalisation of Neukirch's CFT mechanism and explicit higher GCFT by F.

HCFT uses Milnor K_n-groups or even better their quotients $K_n^t = K_n / \cap_{m \geq 1} mK_n$

One of key difficulties: for a finite Galois extension L/F of higher fields the homomorphism $K_n(F) \to K_n(L)^{G(L/F)}$ is in general neither injective nor surjective.

I.e., no Galois descent in HCFT.

2d reciprocity map

$$K_2^t(\mathbf{A})/(K_2^t(\mathbf{B}) + K_2^t(\mathbf{C})) \longrightarrow G_K^{\mathrm{ab}}$$

Remark. Unlike 1d CFT, where geometry and arithmetic are essentially the same, HCFT is separated from various geometrical issues.

All known HCFT are GCFT.

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Early work in anabelian geometry used CFT (or closely related theories) in
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It studies certain deformations and bounds on them of certain scheme theoretical objects using anabelian geometry, mono-anabelian transport, theta- and log-links and symmetry and coricity of étale-like objects.

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IUT uses the computation of the Brauer group of a local field but not local CFT and not global CFT.

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Evaluation of functions at special points in IUT can be viewed as related to a generalisation of SCFT at the level of Kummer theory to all number fields.

Evaluation of functions at special points plays a fundamental role in IUT and in SCFT but not in GCFT.

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Explicit GCFT uses very little of torsion elements, instead working with prime elements wrt $(\deg, v)$.

Problem. Is there a version of IUT which uses less of torsion elements and values of functions at torsion elements?

Other concepts of IUT such as mono-anabelian algorithms, categorical geometry aspects, synchronisations, indeterminacies are already present in explicit GCFT and may find further extensions and developments of CFT and HCFT.

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CFT, anabelian geometry and LC

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However, modern expositions of CFT do not use L-functions.

LC conjecturally classifies (irreducible) linear continuous representations of
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One can view LC as a *linear* theory over abelian CFT. Since it is a representation theory, LC inevitably misses various important features of the full absolute Galois group that are not of linear representation type.

For example, anabelian geometry uses the following two group theoretical properties of the absolute Galois group of a number field or of its nonarchimedean completion: each of its open subgroups is centre-free, each nontrivial normal closed subgroup H of any open subgroup, with the property that H is topologically finitely generated as a group, is open. These properties are not used in LC.

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Currently, the main arithmetic achievements in LC are of *special type only*.

100 years after Takagi's pioneering work in GCFT and 50 years after the beginning of LC we are still awaiting for results of general type in LC.

Most fundamental problems in arithmetic LC remain open.

In particular,
purely local presentation of the local LC is unknown.
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L. Lafforgue proved the equivalence between functoriality in LC and the existence of a certain *non-additive* Fourier transforms satisfying a Poisson formula.

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Can the conjectures in arithmetic LC be fully established remaining inside the linear theory, i.e. representation theory for adelic objects and Galois groups and the one-dimensional CFT?

Can non-linear methods help new fundamental developments in LC?

For example, should one use in LC more information about the absolute Galois group of global and local fields, which cannot be reached via representation theory?

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HAT and IUT

1. Each of HAT and IUT employs two different structures/symmetries, to reflect the 2d nature of the relevant problems and issues, i.e. working with curves over global fields.

2. The use of Milnor K_2 in HCFT has some parallels with the role of tripod in anabelian geometry and IUT.

3. The two symmetries in IUT:

geometric additive $\mathbb{F}_\ell^{\rtimes\pm}$-symmetry and arithmetic multiplicative $\mathbb{F}_\ell^{\circledast}$-symmetry

have a number of features which are reminiscent of geometric additive 2d adelic structure and analytic/arithmetic multiplicative 2d adelic structure in HAT.

4. The theta-link in IUT has analogies with 1d adelic self-duality and 2d analytic adeles duality and with the theta-formula used in the computation of 1d and 2d zeta integrals.

5. Change of coordinates in IUT is similar to change of coordinates in HAT.

6. Similarly to additive $\mathbb{F}_\ell^{\rtimes\pm}$-symmetry in IUT, self-duality of 2d geometric adeles adds another dimension. It is used in adelic study of intersection theory on surfaces and in applications to the BSD conjecture.

However, topological aspects, playing fundamental role in HAT, are less prominent in IUT.

HAT and IUT

1. Each of HAT and IUT employs two different structures/symmetries, to reflect the 2d nature of the relevant problems and issues, i.e. working with curves over global fields.

2. The use of Milnor K_2 in HCFT has some parallels with the role of tripod in anabelian geometry and IUT.

3. The two symmetries in IUT:

geometric additive $\mathbb{F}_\ell^{\rtimes\pm}$-symmetry and arithmetic multiplicative $\mathbb{F}_\ell^{\circledast}$-symmetry

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5. Change of coordinates in IUT is similar to change of coordinates in HAT.

6. Similarly to additive $\mathbb{F}_\ell^{\rtimes\pm}$-symmetry in IUT, self-duality of 2d geometric adeles adds another dimension. It is used in adelic study of intersection theory on surfaces and in applications to the BSD conjecture.

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IUT, Diophantine geometry and arithmetic of elliptic curves

IUT is the first structured conceptual theory of fundamental practical importance for Diophantine geometry.

It is explicit and algorithmic.

It proves inequalities, not equalities.

Explicit estimates in IUT produce proofs of effective versions of Szpiro inequalities:

Szpiro abc inequality over mono-complex fields and

Szpiro inequality for Frey elliptic curves over mono-complex fields

These effective versions will have applications to various classes of Diophantine equations, proving the absence of non-trivial integer solutions for large values of parameters. For example,

$$x^p + y^p = z^p, \quad x^p + y^p = az^q, \quad x^p + y^q = z^l.$$

When applying effective inequalities, one may need some lower bound estimates on potential solutions, obtained using classical algebraic number theory.

Existing 'classical' methods can help establish lower bounds for the first and some of the second equations.
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For many classes of Diophantine equations a synergetic work in IUT, algebraic number theory and computational number theory can bring fruitful outcomes.

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IUT and analytic number theory

While not involving a substantial portion of real or complex analysis, outcomes of IUT include fundamental applications in analytic number theory: the abc inequalities and related properties of zeta- and L-functions.

Aspects of IUT shift the study from archimedean places to non-archimedean places and produces applications to the former from the study of the latter.

An interesting challenge is to explore whether the study and use of non-archimedean Gaussians in IUT and other aspects of IUT can be extended to archimedean Gaussians and a direct study of zeta- and L-functions.

Applications of enhanced versions of IUT to central problems in analytic number theory such as Siegel zeros is a recent important development.

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IUT and quantum computing

There is some quantum mechanics 'feel' in some aspects of IUT.

IUT has some analogies to Kodaira–Spencer theory and the latter plays a role in quantum field theory as a string field theory of B–model.

Interaction of frobenius-like and étale-like structures via the Kummer map may be sometimes viewed a little analogous to the relation between particles and waves in quantum mechanics.

Zero-mass objects/non-zero mass objects are compared in [Alien] to étale-like/frobenius-like objects.

The fact that in IUT it is only when one obtains a formal subquotient that forms a "closed loop" then one may pass from subquotient to a set-theoretic subquotient by taking the log-volume is a little similar to a measurement of a quantum system when the quantum wave 'collapses'.

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IUT and quantum computing

Very recently, analogies between ideas and some objects of quantum computing and ideas and some objects of IUT were observed.

One of the key issues for quantum algorithms is whether they can run in polynomial time, instead of exponential time. Controlling loss of information/error correction is crucial.

One of the main mechanisms of IUT for certain hyperbolic curves is how to produce bounds on change of relevant data passing through the theta-link.

The abc inequality that IUT implies can be compared to naively deduced versions of the abc inequality as polynomial versus exponential.

The polynomial versus exponential time issue in quantum computing is also reminiscent of the closed loop issues when working with the log-theta lattice in IUT and the key issue in p-adic Teichmüller theory of whether the p-curvature of a crystal is nilpotent.

Clifford groups in quantum circuits simulate some highly entangled many-body states on classical computers in polynomial time (Gottesman–Knill, Aaronson–Gottesman).

Clifford groups are stabiliser groups.

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Clifford groups are associated with quadratic forms and quadratic aspects are fundamental in IUT (theta symmetries,..., multiradiality).

The topological groups showing up in IUT are typically profinite or close to them.

They are centre free, while the centre of a Clifford group is infinite and the group mod it is finite.

However, when one works with those arithmetic fundamental groups, often one considers them as the projective limit of their quotients which are extensions of a finite group by infinite abelian and such quotients mod their centre are finite groups as well.

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